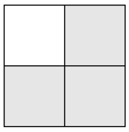
3501. "A y intercept of $y = \mathrm{g}(x)$ must be a y intercept of $y^2 = \mathrm{g}(x)$." True or false?

3502. Evaluate $\displaystyle\lim_{x\to 0.2} \frac{15x^3 + 2x^2 + 4x - 1}{5x^2 + 4x - 1}$.

3503. The four regions of the schematic map below are shaded, or left blank, at random. In the example, three have been shaded.



The probability that at least two shaded regions share a border is designated p. Show that $p = \frac{9}{16}$.

3504. Either prove or disprove the following statement: "If two cubic graphs of the form $y = \mathrm{f}(x)$ have three distinct points in common, then they must be the same cubic graph."

3505. Determine whether each of the following, in which x and y are variables, can be factorised:

(a) $x^2 + y^2$,
(b) $x^2 + xy + y^2$,
(c) $x^2 + 2xy + y^2$.

3506. Show that no line of the form $y = x + k$ is tangent to the curve $x - y + \sqrt{x + y} = 0$.

3507. Two models are proposed to describe a particle's motion: $x = t^3 + t$ and $x = 12t^2 - 47t + 64$.

(a) Find the time T at which these models predict the same position for the particle.
(b) Show that, at time T, both models predict matching values for velocity and acceleration.
(c) Let d be the displacement for $t \in [T, T + 10]$. Find the difference in the value of d predicted by the two models.

3508. Disprove the statement "Every set of three linear simultaneous equations in three unknowns has at least one solution."

3509. A rectangle, with dimensions $1 \times k$, has a circle circumscribed on it.

(a) Find the ratio of areas of circle and rectangle.
(b) Prove that this ratio is minimised for $k = 1$.

3510. Show that the following function is invertible, and give a formal definition of its inverse:

$$\mathrm{f} : \begin{cases} (-1, \infty) \longmapsto \mathbb{R}, \\ x \longmapsto \ln \dfrac{1}{1+x}. \end{cases}$$

3511. Solve for x in $\displaystyle\sum_{r=1}^{\infty} x^{r-1}(1 + x^2) = x + 2$.

3512. The probabilities of events A_i, for $i \in \{1, 2, 3, 4\}$, are given by $\mathbb{P}(A_i) = (1/2)^i$. The four events A_i are all independent of one another.

(a) Find $\mathbb{P}(A_1 \cup A_2)$.
(b) Determine the probability that at least one of the events occurs.

3513. State, with a reason, the general formula for

$$\int \frac{\mathrm{g}'(x)}{\mathrm{g}(x)}\, dx.$$

3514. A function of the form $\mathrm{f}(x) = A \sin kx + B \cos kx$ has $\mathrm{f}(0) = \mathrm{f}''(0) = 0$ and $\mathrm{f}'(0) = 1$. Find, in terms of k, for $x > 0$,

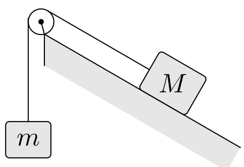
(a) the first x intercept of $y = \mathrm{f}(x)$,
(b) the first turning point of $y = \mathrm{f}(x)$.

3515. The second, fourth and sixth terms of a GP are given as $2x - 3$, x, $2x + 3$. Show that there are four possible values for the common ratio.

3516. The notation $\mathrm{f}^{(n)}(x)$ means the nth derivative of $\mathrm{f}(x)$. Show that the following implication holds:

$$\mathrm{f}^{(k)}(x) = (x + k)e^x \implies \mathrm{f}^{(k+1)}(x) = (x + k + 1)e^x.$$

3517. Two masses are connected by a light, inextensible string, which passes over a smooth, light pulley as shown in the diagram. Between the pulley and the mass marked M, the string runs along the line of greatest slope. The slope, which is inclined at an angle θ above the horizontal, is modelled as rough, with coefficient of friction μ.



Show that, if M is large enough, the acceleration a ms^{-2} of the system is given by

$$a = \frac{Mg \sin\theta - \mu Mg \cos\theta - mg}{M + m}.$$

3518. As transformations, the graphs $y = \mathrm{f}(x - 2)$ and $y = \mathrm{f}(x) + 2$ are translations of $y = \mathrm{f}(x)$ by 2 units in the **positive** x and **positive** y directions. There is an apparent $\pm$ asymmetry in the process. Explain why there is, in fact, full symmetry.

3519. A regular hexagon has two adjacent vertices at $(1, 2)$ and $(8, 5)$. Show that its area is $87\sqrt{3}$.

3520. A pair of simultaneous equations is given as

$$\ln(x+y) + \ln(x-y) = 2,$$
$$x + y^2 = e.$$

(a) With reference to the first equation, explain why x must be positive.
(b) Solve the simultaneous equations.

3521. Functions f and g are both defined over $\mathbb{R}$ and have range $[0, 1]$. State, with a reason, whether each of the following is necessarily true:

(a) fg has range $[0, 1]$ over $\mathbb{R}$,
(b) $\text{fg}(x) \in [0, 1]$ for all $x \in \mathbb{R}$.

3522. The parabola $y = -x^2 + ax + b$ has a maximum at point (p, q). Find the equation of the monic parabola which has a minimum at point $(p, -q)$.

3523. Prove that, for $X \sim N(\mu, \sigma^2)$ and $k > 0$,

$$\mathbb{P}\left(X^2 > kX\right) = \tfrac{1}{2} + \mathbb{P}\left(X > k\right).$$

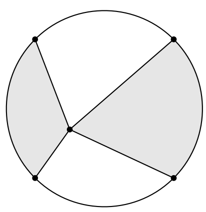
3524. An object is projected over flat ground at speed u. Prove that the maximum range is given by

$$d_{\max} = \frac{u^2}{g}.$$

3525. True or false?

(a) If $\sqrt{a} = \sqrt{b}$, then $a = b$.
(b) If $a = b$, then $\sqrt{a} = \sqrt{b}$.

3526. From a point P inside a circle, four regions are set up, bounded by four arcs of the same length and four straight lines, as shown. Two opposite regions are shaded. None of the angles at P are reflex.



Prove that half of the circle is shaded.

3527. Show that the normal to the curve $y = xe^x$ at the point $x = -2$ has equation

$$e^2 y = e^4 x + 2e^4 - 2.$$

3528. You are given that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, and also that $p\mathbf{a} + q\mathbf{b}$ and $-q\mathbf{a} + p\mathbf{b}$ are perpendicular, for some non-zero p, q. Prove that $|\mathbf{a}| = |\mathbf{b}|$.

3529. Two red, two green and two blue counters are put in a bag. Three counters are then drawn out at random, without replacement. Find

(a) $\mathbb{P}$(RGB in that order),
(b) $\mathbb{P}$(RGB in any order).

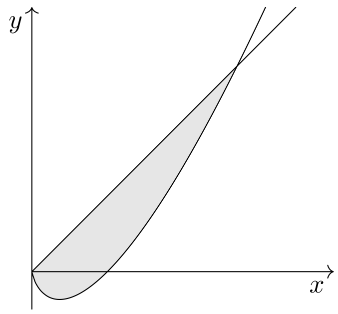
3530. A curve is given as

$$x^2 y^3 - 2x = 3y.$$

Show that every line $y = k$, with one exception to be determined, intersects the curve exactly twice.

3531. Using the substitution $u = e^x$, find $\displaystyle\int \frac{e^x + 1}{e^x + 4}\, dx$.

3532. The diagram shows the curve $y = x \ln x$ and the line $y = x$, and a region enclosed by them.



Determine the exact area of the shaded region.

3533. Show that neither of the following functions has any fixed points:

(a) $\text{f}(x) = x + \dfrac{1}{x^2 + 1}$,
(b) $\text{g}(x) = x^4 - x^2 + x + 1$.

3534. A function is defined over the real numbers as

$$\text{f}(x) = (x^2 - 2x - 8)e^{2x-3}.$$

Solve the inequality $\text{f}(x) > 0$, giving your answer in set notation.

3535. (a) Using the quotient rule, prove that

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

(b) Hence, find $\displaystyle\int \frac{\sec^2 x}{\tan x}\, dx$.

3536. Sketch a counterexample to the following claim: "If a function f is concave over its entire domain, then the graph $y = \text{f}(x)$ can have at most one stationary point."

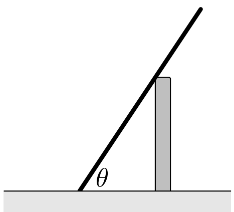
3537. A biologist is researching fungi on the floor of the Congo rainforest. She is trying to discover whether the prevalence of a certain fungus has changed. Historical data suggests that, in any given square metre of forest floor, there is a 28% probability of finding the fungus. She chooses 40 square metres at random to conduct a test.

(a) Write down hypotheses for the test.

(b) 19 of 40 locations show evidence of the fungus. Stating your conclusions carefully, perform the test at the 1% significance level.

3538. Show carefully that the equation ${}^n\mathrm{C}_3 + {}^n\mathrm{C}_4 = {}^n\mathrm{C}_5$ has an empty solution set.

3539. A ladder of mass m is placed against a low wall. It rests in equilibrium with its top protruding. The ground is modelled as smooth, and the wall as rough, with coefficient of friction μ.



(a) Draw a force diagram for the ladder.

(b) Show that the reaction and friction forces at the wall are given by

$$R_{\text{wall}} = \cos\theta(mg - R_{\text{ground}}),$$
$$F_{\text{wall}} = \sin\theta(mg - R_{\text{ground}}).$$

(c) Hence, show that $\mu \geq \tan\theta$.

3540. The equation $\sin\left(x^3\right) = 0$ is given. The sequence $\{x_1, x_2, ...\}$ is defined by the positive roots of the equation, listed in ascending order. Prove that the sequence diverges.

3541. Sketch $x^3y^2 = 1$.

3542. This question concerns the definite integral

$$I = \int_0^1 \arcsin x \, dx.$$

(a) Sketch the area represented by this integral.

(b) Show that $I = \frac{1}{2}\pi - \int_0^{\frac{\pi}{2}} \sin y \, dy$.

(c) Hence, determine the value of I.

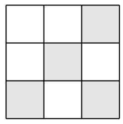
3543. Show that, in a cube, the acute angle between an edge and a space diagonal is $\arccos \frac{1}{\sqrt{3}}$.

3544. At time $t = 0$, a particle of mass m kg is at rest at the origin. A force $F = m \sin 3t$ N then acts on the particle, for $t \geq 0$.

(a) Show that $x = -\frac{1}{9}\sin 3t + \frac{1}{3}t$ metres.

(b) Find the maximum speed during the motion.

3545. In the three-by-three grid below, four squares have been shaded, none of which share an edge:



Prove that there are six ways of doing this.

3546. Region R is defined as satisfying both of

$$x^2 + y^2 < 36,$$
$$(x-6)^2 + y^2 > 36.$$

Determine the exact area of region R.

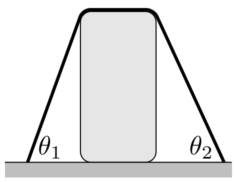
3547. By rewriting a^x over base e, show that

$$\int a^x \, dx = \frac{a^x}{\ln a} + c.$$

3548. Line $y = 9x + 5$ is tangent to curve $y = x^3 - 3x^2$ at P, and also intersects it at another point Q. Determine the coordinates of P and Q.

3549. Solve $\sin\theta(60\sin\theta + 29) = 12$, for $\theta \in [0, 360°)$.

3550. A load of mass m kg is tied down, in equilibrium, in the back of a delivery truck. A single rope is used. The sections of rope on either side of the load are at different angles to the horizontal: $\theta_1 \neq \theta_2$. The tensions either side of the load are $1.7mg$ and $2.5mg$ N; they exert the same vertical component of $1.5mg$ N downwards on the load.



(a) Explain, with reference to tensions, how you know that none of the contacts can be smooth.

(b) The truck floor exerts a contact force on the base of the load. Determine the magnitude of the horizontal and vertical components of this contact force.

3551. A graph is defined by the equation

$$x^3y + x(y-1) = 0.$$

Show that the graph has points of inflection with y coordinate $\frac{3}{4}$.

3552. The random variable X has distribution $N(\mu, \sigma^2)$. It is given that

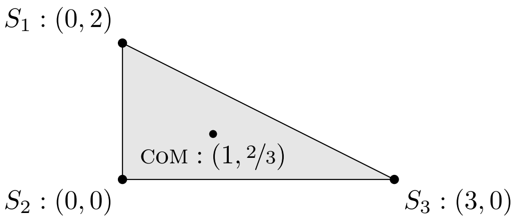
$$\mathbb{P}\big(X \in [0,1]\big) = \mathbb{P}\big(X \in [1,2]\big) = \tfrac{1}{4}.$$

Find σ.

3553. (a) Show that, for small λ, $\sqrt{9 - 24\lambda} \approx 3 - 4\lambda$.

(b) A pair of graphs are given by $y = \lambda x^2 - x + 7$ and $y = 2x + 1$, where λ is a constant close to zero. Show that the graphs intersect at $x \approx 2$ and $x \approx \frac{3}{\lambda} - 2$.

3554. A triangular stage, of weight W, is set up for a pub gig. The locations of the supports and the centre of mass are shown below, in **plan** view, with a coordinate system in metres.



(a) By considering moments about the x axis of the diagram, determine the reaction force R_1 at S_1, in terms of W.

(b) Find R_2 and R_3 in terms of W.

3555. In an (x, y) plane, two unit circles have equations

$$x^2 + y^2 = 1,$$
$$(x-4)^2 + (y-2)^2 = 1.$$

Find the equations of the two parallel lines that are tangent to both circles.

3556. A manager, two deputies and three secretaries are to be chosen from a group of twenty-five people. Determine the number of different ways in which this can be done.

3557. From first principles, prove that

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}+1}\right) = \frac{-1}{2(\sqrt{x}+1)^2\sqrt{x}}.$$

3558. A function is defined over $\mathbb{R}$ as

$$\mathrm{f}(x) = \sin x + \sqrt{3}\cos x.$$

Show that the best linear approximation to $\mathrm{f}(x)$ at $x = \pi$ is $\mathrm{g}(x) = -x + \pi - \sqrt{3}$.

3559. A family of loci is defined by $\ln x - \ln y = k$, for $k \in \mathbb{R}$. On a set of (x, y) axes, shade the set of points which lie on at least one of the loci.

3560. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a polynomial function f:

① $\mathrm{f}(x)$ has a factor of $(x-p)^2$,

② $y = \mathrm{f}(x)$ has a stationary point at $x = p$.

3561. Solve $\displaystyle\sum_{r=1}^{3} \frac{1}{2x - 2r + 5} = 0$.

3562. Show that $y = \operatorname{arccot} x$ has a point of inflection, and find its coordinates.

3563. A parametrically defined graph has x and y as functions of t. The following equations hold:

$$\tfrac{d}{dt}(2x + y) = 1,$$
$$\tfrac{d}{dt}(x - y) = 0.$$

At parameter $t = 3$, the graph is at $(5, 0)$. Find x and y in terms of t.

3564. Find A, B, C in the following identity:

$$\frac{Ax + B}{x^2 + x + 1} + Cx - 1 \equiv \frac{6x^2 + 5x + 1}{x^3 - 1}.$$

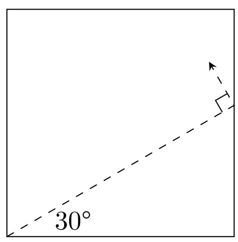
3565. This question concerns Euclid's proof that there are infinitely many prime numbers. Assume, for a contradiction, that there are finitely many prime numbers, written in a list as $p_1, p_2, ..., p_n$.

(a) Show that $P = p_1 p_2 ... p_n + 1$ is prime.

(b) Show that P is not in the original list.

(c) Hence, prove that there are infinitely many prime numbers.

3566. A nanobot is programmed to turn 90° when it hits a wall. It is released inside a box a metre square, from one of the corners, at an angle of 30° from one of the walls.



Show that, at its third collision with a wall, the displacement of the nanobot is

$$s = \tfrac{4}{3} - \tfrac{4\sqrt{3}}{9} \text{ m}.$$

3567. A particle is projected from the point $(10, 0)$ with speed $28\sqrt{2}$ at 45° above the horizontal. Show that the equation of the trajectory is

$$160y = -x^2 + 180x - 1700.$$

3568. Prove that the quintic curve $y = x(x^2 - 1)^2$ has rotational symmetry around the origin.

3569. A hand of four cards is dealt from a standard deck. Find the probability that three are of one suit, and one is of another.

3570. A cubic is defined, for constants a, b, c, d, as

$$\text{f}(x) = ax^3 + bx^2 + cx + d.$$

Prove that there is exists some $p \in \mathbb{R}$ for which the quantity $\text{f}(p - k) + \text{f}(p + k)$ is independent of k.

3571. One of the following statements is true; the other is not. Identify and disprove the false statement.

① $\sin^2 x - 2\sin x = 0 \implies \sin x = 0,$

② $2\sin^2 x - \sin x = 0 \implies \sin x = 0.$

3572. Solve the equation $x^{\frac{3}{5}} - 2x^{\frac{2}{5}} - 15x^{\frac{1}{5}} = 0$.

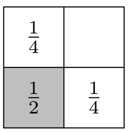
3573. This question concerns the mean value $\bar{Q}$ of the quantity $Q = x + y$, for points (x, y) on the unit circle $x^2 + y^2 = 1$.

(a) Explain why $\bar{Q} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \cos\theta + \sin\theta \, d\theta$.

(b) Find $\bar{Q}$.

3574. Find and classify all stationary points of the curve $y = \sin^3 x + \cos 2x$, and sketch it for $x \in [0, 2\pi]$.

3575. A computer programmer is modelling a random walk on a 2×2 grid. At any iteration, one square of the four is shaded. The following iteration, the probabilities, relative to the current position, are as shown in the diagram.



Find the probability that, after three iterations, the shaded square is where it began.

3576. An equilateral triangle is drawn in an (x, y) plane, in the positive quadrant. Its base lies on the x axis, occupying the interval $\{x \in \mathbb{R} : 0 \le x \le 1\}$.

(a) Find the coordinates of the third vertex, and verify that $y = \sqrt{3}x$ is the equation of one of the sides.

(b) Find the equation of the angle bisector at O.

(c) Hence, or otherwise, prove that the centre of the triangle divides its height in the ratio $1 : 2$.

3577. Three dice have been rolled, giving scores X, Y, Z. Given that $XYZ = 12$, find the probability that $X + Y + Z = 7$.

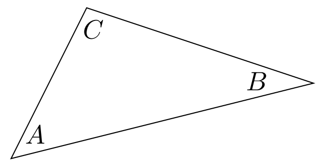
3578. A curve is given as $y = \dfrac{e^x}{x^2 + e^x}$.

(a) Show that the curve has SPs at $x = 0, 2$.

(b) Show that, as $x \to \pm\infty$, $y \to \frac{1}{2} \pm \frac{1}{2}$.

(c) You are given that the SP at $x = 0$ is a local maximum, and at $x = 2$ is a local minimum. Sketch the curve.

3579. Let A, B, C be interior angles of a triangle.



(a) Show that $\tan C = \dfrac{\tan A + \tan B}{\tan A \tan B - 1}$.

(b) Hence, show that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

3580. A statistician is studying orangutans in the wild. He suggests that their masses can be modelled by $M \sim N(60, 12^2)$, in units of kilograms.

(a) Using this model, find the probability that

i. any orangutan will weigh at least 70 kg,

ii. a group of five orangutans will weigh, on average, at least 70 kg.

(b) Explain why neither value will be useful to a biologist studying orangutans.

3581. Show that the graph $\sqrt{y - 1} = \cos x$ is formed of distinct sections of a sinusoidal wave.

3582. Find the set of values of the constant k such that the function $\text{g}(x) = x^3 + kx^2 + 3x + 4$ is invertible over the domain $\mathbb{R}$.

3583. An attempt is made to solve $\text{f}(x) = x^3 - 2x = 0$ numerically, by the Newton-Raphson method.

(a) Show that tangents to $y = \text{f}(x)$ at $x = 0$ and $x = 1$ intersect the x axis at $x = 1$ and $x = 0$ respectively.

(b) Hence, explain why, if either of these values is taken as a starting point, the Newton-Raphson method will fail to converge to a root.

3584. The interior angles of an n-gon form an AP. Give, in terms of n and in radians, the set of possible values for the largest angle.

3585. A monkey of weight $\frac{9}{10}mg$ is holding onto the long rope, which is light and inextensible, of a rising balloon. The balloon has weight $\frac{1}{10}mg$, and the rope is 2.4 metres long.

Initially, the balloon is rising at a constant speed of 1 ms^{-1} and the monkey is at the bottom of the rope, at rest relative to the balloon.

The monkey then begins to climb, and the tension in the rope increases to $\frac{6}{5}mg$.

(a) Find the accelerations of monkey and balloon.
(b) Find the time taken for the monkey to reach the balloon.

3586. True or false?

(a) $x^2 + y^2 + z^2 = 1$ is a sphere.
(b) $|x| + |y| + |z| = 1$ is a cube.

3587. Simplify $\dfrac{e^{4x+1} - 9e^{-1}}{e^{2x} - 3e^{-1}}$.

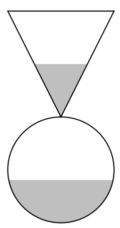
3588. Show that $\displaystyle\int_{-1}^{0} 2\ln(2x+3)\,dx = \ln 27 - 2$.

3589. Statement S is: "If $\mathrm{f}(x)$ is increasing for all $x \in \mathbb{R}$, then the range of f is $\mathbb{R}$." State, with a reason, whether S is true if

(a) f can be any function,
(b) f is a polynomial.

3590. In this question, you should use the fact that the *volume of a spherical cap* is given by $V = \frac{2}{3}\pi r^2 h$, where h is the height of the cap and r the radius of the sphere.

An hourglass consists of water dripping from a cone to a sphere and vice versa. Cone and sphere have the same radius r and the same height $2r$. The volume of water exactly fills the cone.



Show that, when the hourglass is placed as shown, the depth of water in the cone H and in the sphere h are related by $H^3 = 8r^2(r-h)$.

3591. By expressing $\tan 3\theta$ as $\tan(2\theta + \theta)$, prove that

$$\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

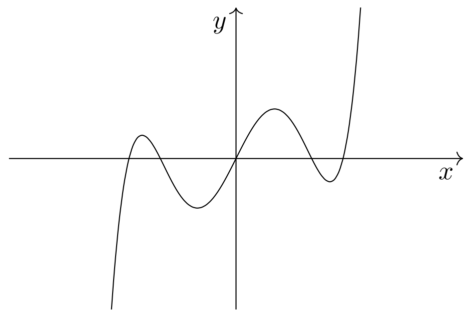
3592. Sketch the graph $y = \dfrac{2}{e^{-x} + e^{x}}$.

3593. A student hypothesises that, if $y = \mathrm{f}(x)$ satisfies the differential equation

$$\frac{dy}{dx} + 3y = 0,$$

then $y = A\,\mathrm{f}(x)$, for $A \in \mathbb{R}$, should also satisfy it. Either prove or disprove this hypothesis.

3594. Let $\mathrm{f}(x)$ be a positive polynomial which contains no even powers of x. This includes no constant term, which may be thought of as a term in x^0. An example graph $y = \mathrm{f}(x)$ is shown below:



(a) Give the single transformation which takes the graph $y = \mathrm{f}(x)$ onto the graph $x = -\mathrm{f}(y)$.
(b) State, with a reason, whether it is certain that the curve $y = \mathrm{f}(x)$ intersects the following:
  i. $x = -\mathrm{f}(y)$,
  ii. $x = -\mathrm{f}(y) + 1$,
  iii. $x = -\mathrm{f}(y+1)$.

3595. You are given that $a\sin x + \cos^2 x = 1$, where a is non-zero and x is not a multiple of π.

(a) Show that $\sin x = a$.
(b) Hence, show that $\tan\frac{1}{2}x = a\sec^2\frac{1}{2}x$.
(c) Hence, show that

$$\tan\tfrac{1}{2}x = \frac{1 \pm \sqrt{1-a^2}}{a}.$$

3596. True or false?

(a) $x^2 - 1 \equiv (x-1)(x+1)$,
(b) $x^3 - 1 \equiv (x-1)(x^2+x+1)$,
(c) $x^4 - 1 \equiv (x-1)(x^3+x^2+x+1)$.

3597. Two particles are moving in one dimension. At $t = 0$, they are a distance of 1 metre apart, and their velocities are subsequently given by

$$v_1 = te^t,$$
$$v_2 = (t-1)e^t.$$

Show that, according to the information given in the question, they either meet at $t = \ln 2$ or they don't meet at all.

3598. Either prove or disprove the following statement: "A polynomial function of odd degree must have at least one fixed point."

3599. A differential equation is given as

$$\frac{dy}{dx} = -5y.$$

(a) Verify that $y = e^{-5x}$ satisfies the DE.

(b) Another solution is proposed: $y = \mathrm{g}(x)e^{-5x}$, for some function g. Show that g is a constant function.

3600. State whether each of the following graphs has at least one line of symmetry:

(a) $y = \sin x + \cos x$,

(b) $y = \arcsin x + \arccos x$.

——— End of 36th Hundred ———